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Irregular Fluctuations in Uncoupled Map Lattices

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Analytic approximations for the spatial average and its variance are derived for a system of N uncoupled chaotic logistic maps with growth parameter $r = 4$. The arising nontrivial closure problem is investigated with various techniques related to the classical moment problem. A Lyapunov-like linear stability analysis is presented for the transient as well as for the fluctuation regime.

KEY WORDS: Uncoupled logistic maps; chaos; fluctuations; theoretical models; computer simulations.

1. INTRODUCTION

The onset of space-time chaos as well as the nature and the origin of accompanying fluctuations in coupled systems has attracted a lot of interest in recent years.(1–3) Though there exist some exact results,(4, 5) most of the work reported so far concentrates on computer experimental studies of spatially extended systems consisting of a huge number of units. In this paper we study a system of *uncoupled* logistic maps which is of interest in its own right. Here the description of large systems is reduced by the introduction of only two global variables, the spatial average and its variance. Like in classical problems of statistical mechanics, i.e., the BBGKY hierarchy,(6) we derive a set of equations linking the spatial average and and its higher order variances. However, since the hierarchy does not obey a closed equation, a nontrivial closure problem arises. In contrast to various classical problems, straightforward truncation schemes completely fail.(7) Hence, the principal objective of this paper is to develop closure schemes at the level of low order variances. Using various approaches—intimately connected with the classical moment problem(8)—higher-order variances are approximated

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by the spatial average and the variance. The system considered is introduced in Section 2, while in Section 3 we present and analyze the nature of the fluctuations of all relevant quantities by large scale computer experiments. Sections 4, 5 and 6 treat the closure problem with various models. In Section 7 we give a linear stability analysis of our models presented. A summary and a future aspect follow in Section 8.

2. PROBLEM FORMULATION

We consider a system of N uncoupled identical iterative maps given by the prescription

$$x_i(t+1) = f(x_i(t)) \qquad i = 1,..., N \tag{2.1}$$

The function $f(x_i)$ is chosen as the logistic map $f(x_i) = rx_i(1-x_i)$, where r defines the growth parameter. Let's define macroscopic variables, the time-dependent moments $\langle x^k \rangle$, by

$$\langle x^k \rangle = \frac{1}{N} \sum_{j=1}^{N} x_j^k(t) \tag{2.2}$$

and the time-dependent variances $\langle d_k \rangle$ by

$$\langle d_k \rangle = \frac{1}{N} \sum_{j=1}^{N} (x_j(t) - \langle x \rangle)^k \tag{2.3}$$

Be $X(t) := \langle x \rangle$ and $d(t) := \langle d_2 \rangle$. For the time evolution of the mean $X(t)$ and for its variance $d(t)$ one finds

$$X' = r[X(1-X) - d] \tag{2.4}$$

$$d' = r^2[d(1-2X)^2 - d^2 + \langle d_4 \rangle - 2\langle d_3 \rangle(1-2X)] \tag{2.5}$$

The prime notation in (2.4) and (2.5), which will be used further, assumes the variable to be evaluated at time $t+1$, otherwise all variables are supposed to be taken at time t.

Equations (2.4) and (2.5) can be regarded as the first two of a hierarchy of a set of N equations. However, the set is not closed since the time evolution of the d_k demands the knowledge of the variances up to d_{2k}. One of the aims of this report is to find reasonable approximations for the higher variances $\langle d_3 \rangle$ and $\langle d_4 \rangle$ in order to close the set of Eqs. (2.4) and (2.5).

If we make a computer simulation with N logistic equations, the interesting quantities are observed in course of time. Starting with an arbitrary initial state, after a transient phase the (smoothed) actual time-dependent distribution $p(x)$ of the N x_i-values produced by the mapping (2.1) comes close to the equilibrium distribution $p_0(x)$ provided that N is sufficiently large, cf. Fig. 2a in Section 3. In the following, we restrict ourselves to the case $r=4$, where $p_0(x)$ is well-known:

$$p_0(x)=\frac{1}{\pi\sqrt{x(1-x)}} \qquad 0\leqslant x\leqslant 1 \tag{2.6}$$

The (smoothed) fluctuating deviation from $p_0(x)$ will be called $\Delta p(x)$ such that the actual distribution $p(x)$ can be written as

$$p(x)=p_0(x)+\Delta p(x) \tag{2.7}$$

The normalization condition $\int_0^1 p(x)\,dx=1$ then demands

$$\int_0^1 \Delta p(x)\,dx=0 \tag{2.8}$$

The fluctuating averages of x^j and d_k can be expresses in terms of the probability distributions in the following way:

$$\langle x^k\rangle=\int_0^1 x^k p(x)\,dx=\langle x^k\rangle_0+\langle x^k\rangle_\Delta=\langle x^k\rangle_0+\Delta(x^k) \tag{2.9}$$

$$\begin{aligned}\langle d_k\rangle&=\int_0^1 (x-\langle x\rangle)^k\,p(x)\,dx=\langle d_k\rangle_0+\Delta d_k\\&=\int_0^1 (x-\langle x\rangle_0)^k\,p_0(x)\,dx+\Delta d_k\end{aligned} \tag{2.10}$$

$\langle\ \rangle$, $\langle\ \rangle_0$ and $\langle\ \rangle_\Delta$ means average using the probability density $p(x)$, $p_0(x)$ and $\Delta p(x)$, respectively. In particular,

$$\langle x\rangle_\Delta=\Delta x, \qquad \langle x^2\rangle_\Delta=\Delta x+\Delta d+(\Delta x)^2 \tag{2.11}$$

$$\langle x^3\rangle_\Delta=\tfrac{9}{8}\,\Delta x+\tfrac{3}{2}\,\Delta d+\Delta d_3+\tfrac{3}{2}(\Delta x)^2+3\,\Delta d\,\Delta x+(\Delta x)^3 \tag{2.12}$$

$$\begin{aligned}\langle x^4\rangle_\Delta=\tfrac{5}{4}\,\Delta x+\tfrac{3}{2}\,\Delta d+2\,\Delta d_3+\Delta d_4+\tfrac{9}{4}(\Delta x)^2\\+6\,\Delta d\,\Delta x+4\,\Delta d_3\,\Delta x+6\,\Delta d(\Delta x)^2+2(\Delta x)^3+(\Delta x)^4\end{aligned} \tag{2.13}$$

For $r = 4$ the asymptotic values are

$$\langle x \rangle_0 = \tfrac{1}{2}, \quad \langle d \rangle_0 = \tfrac{1}{8}, \quad \langle d_3 \rangle_0 = 0, \quad \langle d_4 \rangle_0 = \tfrac{3}{128} \tag{2.14}$$

3. FLUCTUATIONS

We first performed large-scale simulations for systems containing up to 10^7 equations in order to analyze empirically the time evolution of the quantities to be approximated. They all fluctuate around their mean values. The first four deviations Δx, Δd, Δd_3 and Δd_4 with mean value zero are depicted in Fig. 1a and Fig. 1b. We observe that Δd_4 is strongly correlated with Δd, whereas Δd_3 is strongly anticorrelated with Δx. The computer experimental studies reveal clearly that all fluctuations decrease roughly like $1/\sqrt{N}$ with increasing system size N. The corresponding probability distributions are approximatively Gaussian. For N fixed, the absolute values of Δd_k generally decrease with increasing k.

The same observations can be made for the time-dependent probability density $p(x)$ shown in Fig. 2. During the time evolution $p(x)$ fluctuates in a random-like manner around its asymptotic mean value $p_0(x)$. These observations strongly suggest dynamical behaviour indistinguishable from a white noise process for sufficiently large N. Therefore we analyze several single-variable time series of the fluctuations of the first moment $\Delta X = \{\Delta x(0), \Delta x(1),..., \Delta x(T)\}$, which contains information of the whole N-dimensional space. The idea of the existence of a high-dimensional chaotic process in the uncoupled system could be supported by implementing the Grassberger–Procaccia algorithm.[9] Our computer experiments based on embedding dimensions up to eight for large systems of uncoupled cells show clearly that the correlation exponent ν, usually providing a tight lower bound of the fractal dimension D, is always extremely close to the embedding dimension. This is to be expected. The initial conditions are chosen at random such that the dynamical time evolutions of the individual cells—in absence of interactions—are independent. We infer that for sufficiently large N, the fluctuations stem from an N-dimensional chaotic process. The above observations may be contrasted to highly nontrivial fluctuations observed in *coupled* map lattices. There, the mean square deviations from the spatial average do not tend to vanish icon the thermodynamic limit.[2,3] Lack to analytical study, their nature and origin is still a subject of discussions. A rigorous mathematical approch to this delicate problem of nontrivial fluctuations in coupled and uncoupled maps with illuminating examples has recently been presented by Bunimovich and Jiang.[10] Note however that in *weakly* coupled map lattices also Gaussian fluctuations decreasing with the system size N have been reported.[1,3]

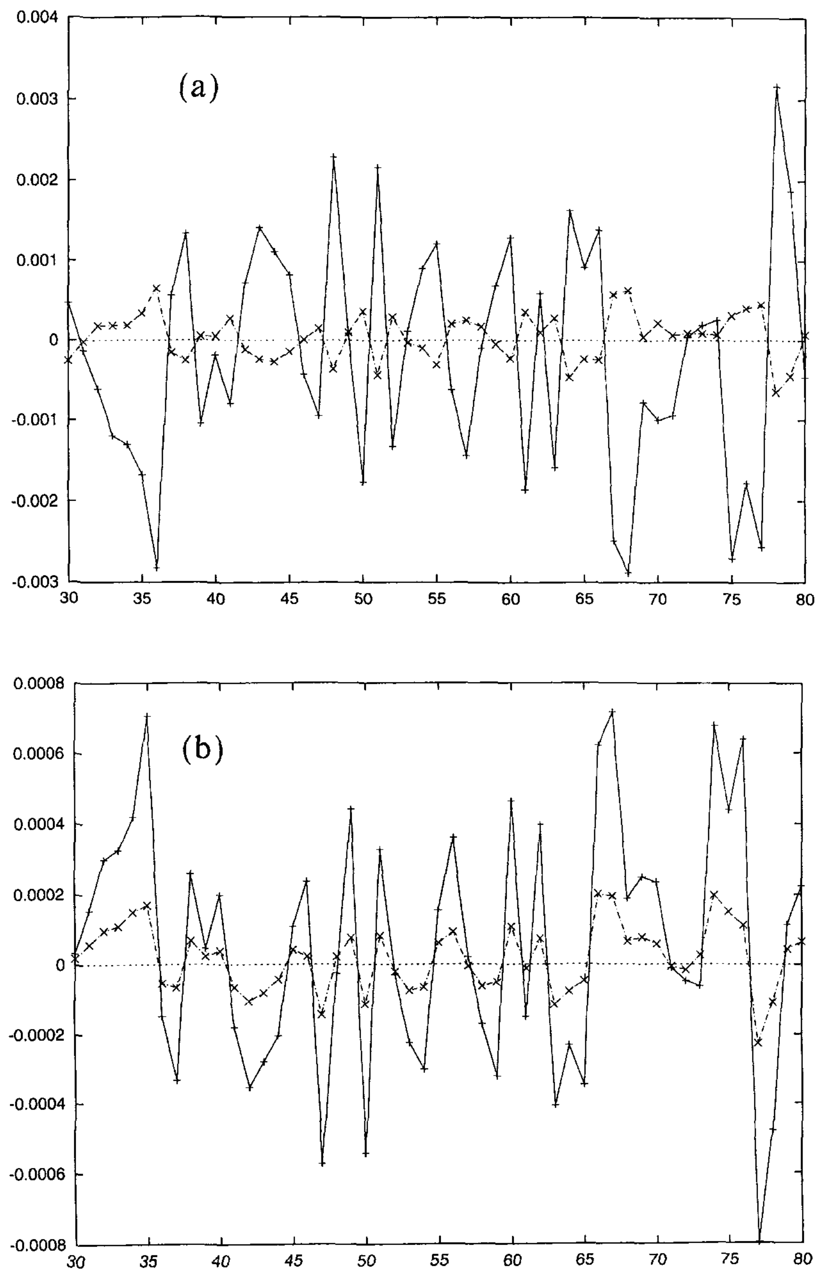
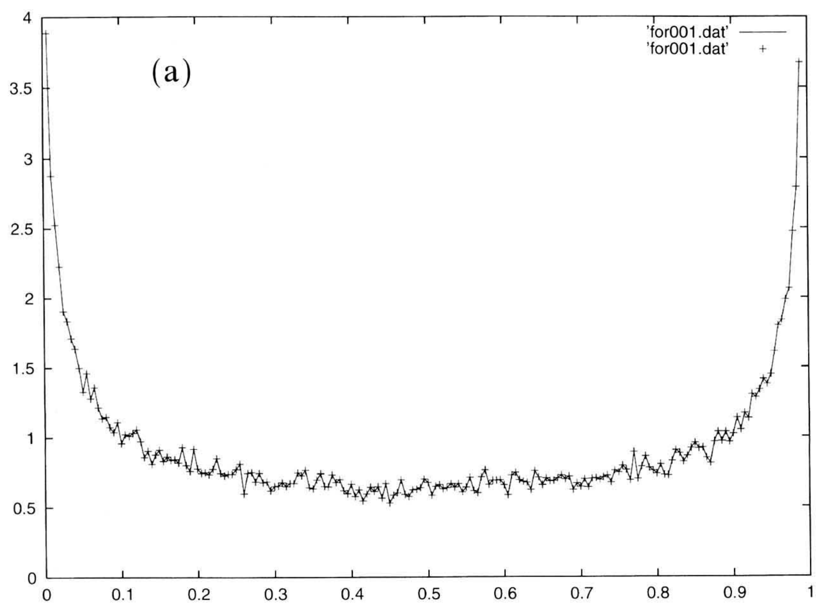


Fig. 1. Time evolution of the deviations for $t = 30,\ldots, 80$ and $N = 2^{16}$: (a) Δd_3 (×) and Δx (+), (b) Δd_4 (×) and Δd (+).



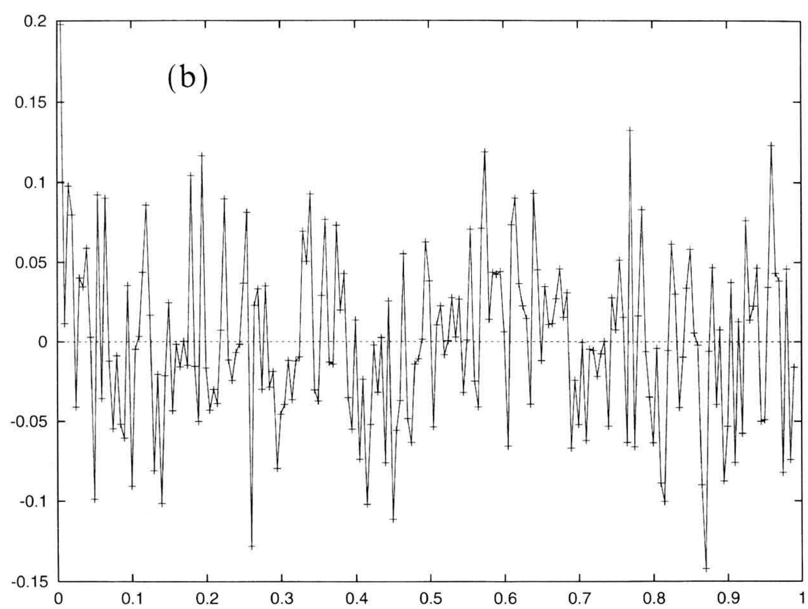


Fig. 2. (a) $p(x)$ at fixed time, (b) $\Delta p(x)$ at fixed time (c) time evolution of $\Delta p(x)$ for fixed $x = 0.0125$, 0.0275 and 0.5025 (from above); $N = 2^{16}$.

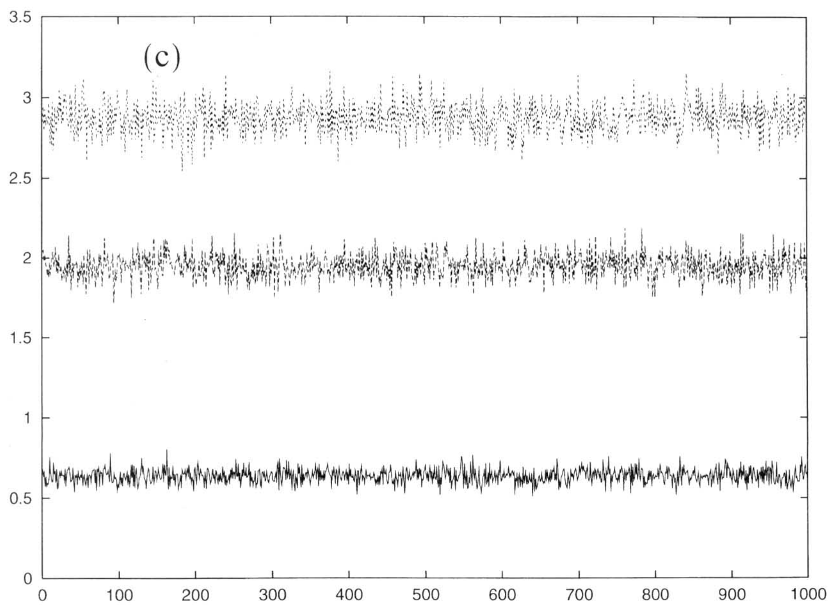


Fig. 2. (*Continued*)

However, in *globally* coupled maps[2] also Gaussian fluctuations have been observed, whose variances saturate at a critical value $N = N_c$.

4. SIMPLE CLOSURE OF THE SET OF TWO EQUATIONS

To simplify our following calculations we transform Eqs. (2.4) and (2.5) into the deviation representation:

$$\Delta x' = -4[\Delta d + (\Delta x)^2] \tag{4.1}$$

$$\Delta d' = 16[\Delta d_4 + 4\,\Delta d_3\,\Delta x - \tfrac{1}{4}\,\Delta d + \tfrac{1}{2}(\Delta x)^2 - (\Delta d)^2 + 4\,\Delta d(\Delta x)^2] \tag{4.2}$$

The simplest model would be to set Δd_3 and Δd_4 equal to zero. It turns out, however, that this approximation is too crude since the iteration of Eqs. (4.1) and (4.2) explodes in this case (see below). Therefore, a refined approximation will be used. Due to the observation made in Fig. 1, we assume that Δd_3 is proportional to Δx and Δd_4 is proportional to Δd. Hence we start with the ansatz

$$\Delta d_3 = \beta_3\,\Delta x \qquad \text{and} \qquad \Delta d_4 = \beta_4\,\Delta d \tag{4.3}$$

Then Eqs. (4.1) and (4.2) result in

$$\Delta x' = -4[\Delta d + (\Delta x)^2] \tag{4.4}$$

$$\Delta d' = 16[(\beta_4 - \tfrac{1}{4})\,\Delta d + (4\beta_3 + \tfrac{1}{2})(\Delta x)^2 - (\Delta d)^2 + 4\,\Delta d(\Delta x)^2] \tag{4.5}$$

Strongly confirmed by various computer experiments Fig. 1 suggests that the fluctuating coefficients β_3 and β_4 are close to $-\frac{1}{5}$ and $+\frac{1}{4}$, respectively. This stands in marked contrast to the simplest model (see above) which corresponds to $\beta_3 = \beta_4 = 0$. Now we assume that β_3 and β_4 are free parameters (fixed, but arbitrary). Then, the set of Eqs. (4.4), (4.5) has the fixed point $(\Delta x, \Delta d) = (0, 0)$ and linear stability analysis leads to the eigenvalues of the Jacobian $\mu_1 = 4(4\beta_4 - 1)$ and $\mu_2 = 0$, the non-trivial eigenvalue μ_1 having its origin in the second basic equation, i.e., (4.5). Hence, the asymptotic behaviour of (4.4) and (4.5) does not depend on β_3, and the fixed point is reached provided that β_4 satisfies $\frac{3}{16} < \beta_4 < \frac{5}{16}$. Note that the naive choice $\beta_4 = 0$ leads to instability. Also the "better" approximation $\beta_4 = (\Delta d_4 / \Delta d) = (\langle d_4 \rangle_0 / \langle d \rangle_0) = \frac{3}{16}$, would not stabilize the system. One can easily show that the dynamics is unstable at the left extreme of its stability domain.

5. CLOSURE BY DENSITY MODELS WITHOUT FREE PARAMETERS

Let's now turn to find models for the deviation $\Delta p(x)$ from the equilibrium density which allows to determine the deviations Δd_3 and Δd_4 as functions of Δx and Δd, see below. We start with the parabolic ansatz:

$$\Delta p(x) = \alpha_0 + \alpha_1(x - 0.5) + \alpha_2(x - 0.5)^2 \tag{5.1}$$

The first and third terms are—like $p(x)$—even functions with respect to $x = 0.5$, while the second term is odd. Given Δx and Δd and using (2.11) together with condition (2.8) we have a system of three linear equations for α_0, α_1 and α_2. The solution is

$$\alpha_0 = -15[\Delta d + (\Delta x)^2], \qquad \alpha_1 = 12\Delta x, \qquad \alpha_2 = 180[\Delta d + (\Delta x)^2] \tag{5.2}$$

Inserting this in Eqs. (2.12) and (2.13) yields Δd_3 and Δd_4 as nonlinear functions of Δx and Δd. This results in a closure of our basic set of two equations without any free parameters:

$$\Delta d_3 = -(9/40)\,\Delta x - 3\,\Delta d\,\Delta x - (\Delta x)^3 \tag{5.3}$$

$$\Delta d_4 = [(3/14)\,\Delta d + (51/140)(\Delta x)^2] + 6\,\Delta d(\Delta x)^2 + 3(\Delta x)^4 \tag{5.4}$$

$$\Delta d_{4,3} = [(3/14)\,\Delta d - (51/28)(\Delta x)^2] - 6\,\Delta d(\Delta x)^2 - (\Delta x)^4 \tag{5.5}$$

where the last quantity defined as

$$\Delta d_{4,3} := \Delta d_4 + 4\,\Delta d_3\,\Delta x \tag{5.6}$$

closes Eq. (4.2). The leading terms are

$$\Delta d_3 = -(9/40)\,\Delta x = -0.2250\,\Delta x, \qquad \Delta d_4 = \Delta d_{4,3} = (3/14)\,\Delta d = 0.2143\,\Delta d \tag{5.7}$$

A more realistic model for $\Delta p(x)$ uses $p_0(x)$ itself, which is the probability for a logistic map to have the value x, see Eq. (2.6). Due to its shape, many of the N equations considered favour x-values close to 0 and 1. We assume that for fixed N the fluctuations $\Delta p(x)$ around equilibrium are proportional to $\sqrt{p_0(x)}$ as a function of x. The ansatz

$$\Delta p(x) = a + bx^{-1/4} + c(1-x)^{-1/4} \tag{5.8}$$

fulfills this condition in both ranges close to 0 and 1 where $p_0(x)$ becomes appreciably large. The assumption that Δp is proportional to $\sqrt{p_0(x)}$ comes from similar arguments as the $1/\sqrt{N}$ law for fluctuations. It is supported by empirical data like in Fig. 2c. There, one can see how the typical variations of the fluctuations depend on x. Direct calculation of the variances for *all* values of x confirm our assumption.

Similar calculations as above yield for the three coefficients

$$a = -(462/5)[\Delta d + (\Delta x)^2] \tag{5.9}$$

$$b = -(21/4)\,\Delta x + (693/20)[\Delta d + (\Delta x)^2] \tag{5.10}$$

$$c = (21/4)\,\Delta x + (693/20)[\Delta d + (\Delta x)^2] \tag{5.11}$$

This results in

$$\Delta d_3 = -(87/440)\,\Delta x - 3\,\Delta d\,\Delta x - (\Delta x)^3 \tag{5.12}$$

$$\Delta d_4 = [(237/950)\,\Delta d + (6669/20900)(\Delta x)^2] + 6\Delta d(\Delta x)^2 + 3(\Delta x)^4] \tag{5.13}$$

$$\Delta d_{4,3} = [(237/950)\,\Delta d - (951/1900)(\Delta x)^2] - 6\,\Delta d(\Delta x)^2 - (\Delta x)^4 \tag{5.14}$$

The leading terms are

$$\begin{aligned} \Delta d_3 &= -(87/440)\,\Delta x = -0.1977\,\Delta x \\ \Delta d_4 &= \Delta d_{4,3} = (237/950)\,\Delta d = 0.2495\,\Delta d \end{aligned} \tag{5.15}$$

We have also considered further three-coefficient models. The results of all these approaches are remarkably similar: The leading terms exhibit Δd_3 being proportional to Δx, Δd_4 and $\Delta d_{4,3}$ being proportional to Δd. In the first case, the constant of proportionality is negative (typically $-\frac{1}{5}$), in the second case it is positive (typically $\frac{1}{4}$). This is consistent with Fig. 1. The important quantity $\Delta d_{4,3}$ can in all cases be parametrized as follows:

$$\Delta d_{4,3} = [\phi + (1/4)]\ \Delta d + [\phi - (1/2)](\Delta x)^2 - 6\ \Delta d(\Delta x)^2 - (\Delta x)^4 \tag{5.16}$$

ϕ being $-1/28$ and $-1/1900$ in the two cases displayed above. Any of the other models yields another constant ϕ. Inserting (5.16) in (4.2) yields

$$\Delta d' = 16\{\phi[\Delta d + (\Delta x)^2] - [\Delta d + (\Delta x)^2]^2\} \tag{5.17}$$

This will be used in Section 7, where the stability of the solution will be studied as a function of ϕ. Here we will only mention that $\phi = 0$ corresponds to superstability.

6. CLOSURE BY DENSITY MODELS WITH FREE PARAMETERS

Now we generalize our first ansatz to a polynomial of degree four:

$$\Delta p(x) = \alpha_0 + \alpha_1(x - 0.5) + \alpha_2(x - 0.5)^2 + \alpha_3(x - 0.5)^3 + \alpha_4(x - 0.5)^4 \tag{6.1}$$

In contrast to (5.1) and (5.2) we have an underdetermined system. We define the ratios Q and R of the odd and even contributions

$$Q := \alpha_3/\alpha_1, \qquad R := \alpha_4/\alpha_2 \tag{6.2}$$

$Q = R = 0$ reduces to the old ansatz without free parameters, introducing Q gives one parameter, further expansion to R two parameters. For instance, a small Q means that the linear term gives the dominant contribution to the odd part of $\Delta p(x)$. If Q is large, the cubic term dominates the linear one. Calculating the coefficients yields

$$\alpha_0 = \frac{-21(20 + 3R)}{2(14 + 3R)} [\Delta d + (\Delta x)^2] \tag{6.3}$$

$$\alpha_1 = \frac{240}{20 + 3Q} \Delta x \tag{6.4}$$

$$\alpha_2 = \frac{2520}{14 + 3R} [\Delta d + (\Delta x)^2] \tag{6.5}$$

$$\alpha_3 = \frac{240Q}{20 + 3Q} \Delta x \tag{6.6}$$

$$\alpha_4 = \frac{2520R}{14 + 3R} [\Delta d + (\Delta x)^2] \tag{6.7}$$

The interesting quantities turn out to be

$$\Delta d_3 = \left[-\frac{11}{56} - \frac{4}{7(20+3Q)} \right] \Delta x - 3\,\Delta x\,\Delta d - (\Delta x)^3 \tag{6.8}$$

$$\Delta d_4 = \left[\frac{7}{30} - \frac{4}{15(14+3R)} \right] \Delta d + \left[\frac{113}{420} + \frac{16}{7(20+3Q)} - \frac{4}{15(14+3R)} \right] (\Delta x)^2 + 6\,\Delta d(\Delta x)^2 + 3(\Delta x)^4 \tag{6.9}$$

$$\Delta d_{4,3} = \left[\frac{7}{30} - \frac{4}{15(14+3R)} \right] \Delta d - \left[\frac{31}{60} + \frac{4}{15(14+3R)} \right] (\Delta x)^2 - 6\,\Delta d(\Delta x)^2 - (\Delta x)^4 \tag{6.10}$$

Δd_3 only depends on parameter Q, Δd_4 on both Q and R as expected. It is interesting to note that $\Delta d_{4,3}$—the relevant quantity for the basic equations—only depends on parameter R. The expression is very similar to (5.16). Indeed, (5.16) and (6.10) may be identified via the relation

$$\phi = -\frac{10+R}{280+60R}, \qquad R = -\frac{10+280\phi}{1+60\phi} \tag{6.11}$$

Now, ϕ can be interpreted in ask new way, i.e., in terms of the parameter R. Different ϕ correspond to different three-coefficient models (see Section 5), but the connection was not evident. Now, we have a more general model containing two free parameters. $\Delta d_{4,3}$ only depends on R. A wrong Q, i.e., a bad estimate of at least one of the odd terms of $\Delta p(x)$, does not influence the closure. It is the even contributions which matter. For instance, $\phi = 0$ (superstability) corresponds to $R = -10$. That means that in our model the term of degree 2, see Eq. (6.1), is overcompensated by the term of degree 4. Knowing R, the even coefficients α_0, α_2 and α_4 can be calculated, see (6.3), (6.5) and (6.7), respectively.

In principle, the situation remains the same if even higher polynomials are considered. Now let's turn to the most general case, where $\Delta p(x)$ is expanded as an infinite power series.

$$\Delta p(x) = \alpha_0 + \alpha_1(x-0.5)\left[1 + \sum_{k=2}^{\infty} Q_{2k-1}(x-0.5)^{2k-1} \right] + \alpha_2(x-0.5)^2 \left[1 + \sum_{k=2}^{\infty} R_{2k}(x-0.5)^{2k} \right] \tag{6.12}$$

$\Delta p(x)$ is split up into the odd and even terms. With the definitions

$$Q^{+}_{2k-1} := Q_{2k-1}/2^{2k-2}, \qquad R^{+}_{2k} := R_{2k}k/[(1+2k)\,2^{2k-2}] \tag{6.13}$$

$$Q^{*} := (3/20)\left\{\left[1 + 5\sum_{k=2}^{\infty} Q^{+}_{2k-1}/(3+2k)\right]\Big/\left[1 + 3\sum_{k=2}^{\infty} Q^{+}_{2k-1}/(1+2k)\right]\right\} \tag{6.14}$$

$$R^{*} := (3/14)\left[1 + 21\sum_{k=2}^{\infty} R^{+}_{2k}/(5+2k)\right]\Big/\left[1 + 15\sum_{k=2}^{\infty} R^{+}_{2k}/(3+2k)\right] \tag{6.15}$$

we get

$$\Delta d_3 = \Delta x[-(3/8) + Q^{*}] - 3\,\Delta x\,\Delta d - (\Delta x)^3 \tag{6.16}$$

$$\Delta d_4 = R^{*}\,\Delta d + (\Delta x)^2\,[(3/4) - 4Q^{*} + R^{*}] + 6\,\Delta d(\Delta x)^2 + 3(\Delta x)^4 \tag{6.17}$$

$$\Delta d_{4,3} = R^{*}\,\Delta d + (\Delta x)^2\,[-(3/4) + R^{*}] - 6\,\Delta d(\Delta x)^2 - (\Delta x)^4 \tag{6.18}$$

Again, $\Delta d_{4,3}$ only depends on the even terms and only includes one parameter R^{*}. The odd terms play no role at all. Since many combinations of R_{2k} result in the same R^{*}, it is not possible to find out the even coefficients in a unique way when starting from R^{*}. Looking at the system from an other point of view, R^{*} may also be determined via Eq. (6.18) using the computer experimental Δx, Δd and $\Delta d_{4,3}$ at each time step. Then it comes out as a fluctuating quantity reflecting changes in the shape of the even part of $\Delta p(x)$.

7. STABILITY CONSIDERATIONS AND LYAPUNOV-LIKE EXPONENTS

We come back to the presentation in terms of ϕ. Then the basic two equations are

$$\Delta x' = -4[\Delta d + (\Delta x)^2] \tag{7.1}$$

$$\Delta d' = 16\{\phi[\Delta d + (\Delta x)^2] - [\Delta d + (\Delta x)^2]^2\} \tag{7.2}$$

cf. Eq. (4.1) and (5.17). Now ϕ is again interpreted as a free parameter (arbitrary, but fixed). Linear stability analysis around equilibrium ($\Delta x = \Delta d = 0$) yields the following two solutions:

(i) eigenvalue $\mu_1 = 16\phi$, eigenvector $(\Delta x, \Delta d) = (1, -4\phi)$

(ii) the trivial solution, i.e., eigenvalue $\mu_2 = 0$ with the eigenvector $(1, 0)$.

The limits of stability are $\phi = \pm\frac{1}{16}$, superstability corresponds to $\phi = 0$. In our simplest model (Section 4), we set Δd_3 and Δd_4 equal to zero, thus $\Delta d_{4,3}$ also being vanishing. Retaining only the linear terms in Eq. (5.16) corresponds to $\phi = -\frac{1}{4}$, which is in the unstable region. The two models exhibited in Section 5 lie in the stable region, ϕ being $-\frac{1}{28}$ and $-\frac{1}{1900}$, respectively.

Strictly speaking, the set of Eqs. (7.1), (7.2) is only closed if ϕ is set constant. On the other hand, we may determine a fluctuating ϕ from Eq. (5.16) or (5.17) $=$(7.2) when using the computer experimental Δx, Δd and $\Delta d_{4,3}$ in the course of time. First we stick to (5.16). $\Delta d_{4,3}$ fluctuates as a function of t even for constant ϕ. Only for $\phi = -\frac{1}{4}$, $\Delta d_{4,3} = 0$ in the linear approximation, see above. Figure 3 displays a short part of a computer simulation for $N = 2^{10}$, showing the empirical $\Delta d_{4,3}$ together with the cases $\phi = -\frac{1}{16}$, 0, $\frac{1}{16}$ and $\phi = -\frac{1}{4}$. One can see that the computer experimental curve is typically in the range of the two theoretical ones with $\phi = \pm\frac{1}{16}$. The empiricial ϕ is not constant but fluctuating. A closer look on the whole computer runs reveals that 50% of the empirical values lie inside and outside the "band" of $|\phi| = \frac{1}{16}$, respectively. Thus the median of $|\phi|$ is $\frac{1}{16}$, which

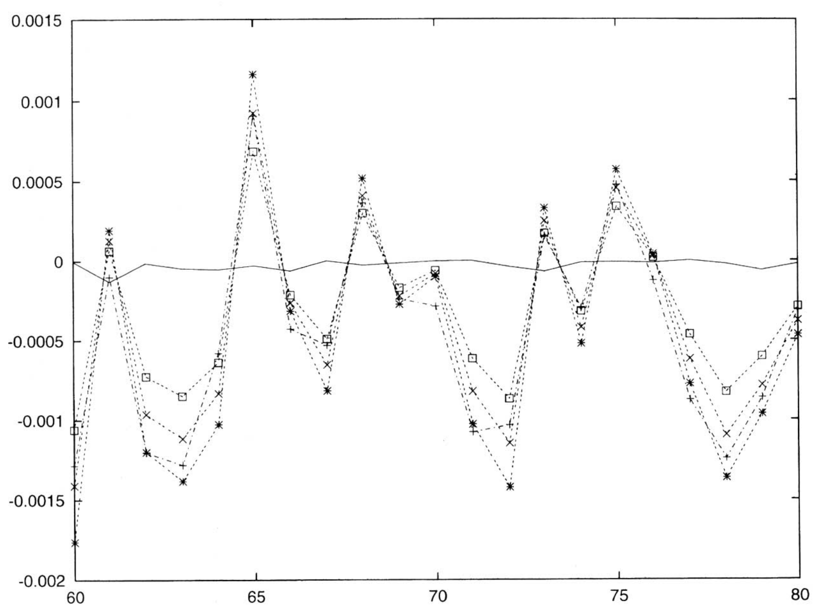


Fig. 3. Time evolution of the empirical $\Delta d_{4,3}$ (+) and for $\phi = 0$ (×), $\phi = -\frac{1}{16}$ (∗), $\phi = +\frac{1}{16}$ (□), $\phi = -\frac{1}{4}$ (full line); $N = 2^{10}$.

corresponds to the two extremes of the stability domain (marginal stability). This is consistent with the fact that the fluctuation of, e.g., $\Delta d_{4,3}$ remains typically the same during the computer experiment run, neither increasing nor decreasing exponentially. Since the distribution of ϕ turns out to be symmetric with respect to zero, $\phi = 0$ is the best constant estimate, if ϕ instead of $|\phi|$ is considered. This is supported by the calculation of the mean-square deviation of $\Delta d_{4,3}(\phi = 0)$, $\Delta d_{4,3}(\phi = \frac{1}{16})$ and $\Delta d_{4,3}(\phi = -\frac{1}{16})$. It turns out that the mean-square deviation with respect to the superstable case ($\phi = 0$) is somewhat (about 50%) smaller compared to that of the left and right extreme of the stability region. For the stability consideration, however, $|\phi|$ is relevant, see also below.

It should be mentioned that the case $\phi = -\frac{1}{4}$ is also included in Fig. 3. As shown above, this results in $\Delta d_{4,3} = 0$ in the linear approximation. Indeed, $\Delta d_{4,3}$ is very close to zero although $N = 2^{10}$ is not too large.

Now we come back to (7.1) and (7.2). By summing up the square of the first equation and the second one we get the simple expression

$$[\Delta d + (\Delta x)^2]' = 16\phi[\Delta d + (\Delta x)^2] \tag{7.3}$$

Calling the quantity in square brackets Δu and displaying the equation as the step from time t to $t+1$, it follows that

$$\Delta u(t+1) = 16\phi_t \, \Delta u(t) \tag{7.4}$$

This linear equation comes out directly and not by the help of a linearization procedure, resulting in the eigenvalue $\mu_1 = 16\phi$. Considering more steps yields

$$|\Delta u(T)| = \left\{ \prod_{t=0}^{T-1} (16\,|\phi_t|) \right\} |\Delta u(0)| \tag{7.5}$$

The geometric mean of the absolute eigenvalues after T steps is $16[\prod_{t=0}^{T-1} |\phi_t|]^{1/T}$. Now we define $\lambda(t)$ in a Lyapunov-like way,

$$|\Delta u(t+1)| = \exp(\lambda(t)) \, |\Delta u(t)| \tag{7.6}$$

After T steps

$$|\Delta u(T)| = \exp(T\lambda_T) \, |\Delta u(0)| \tag{7.7}$$

where λ_T, the arithmetic average of $\lambda(t)$, comes out as

$$\lambda_T = \frac{1}{T} \ln \frac{|\Delta u(T)|}{|\Delta u(0)|} = \ln \left[\left(\prod_{t=0}^{T-1} |16\phi_t| \right)^{1/T} \right] \tag{7.8}$$

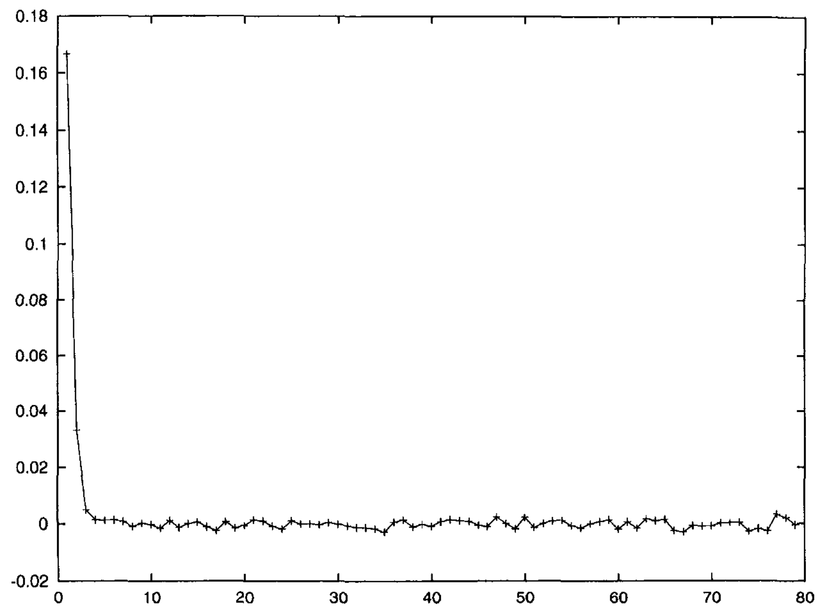


Fig. 4. Time series of $\Delta u(t)$ for $t=1, 2, \ldots, 80$; $N=2^{16}$.

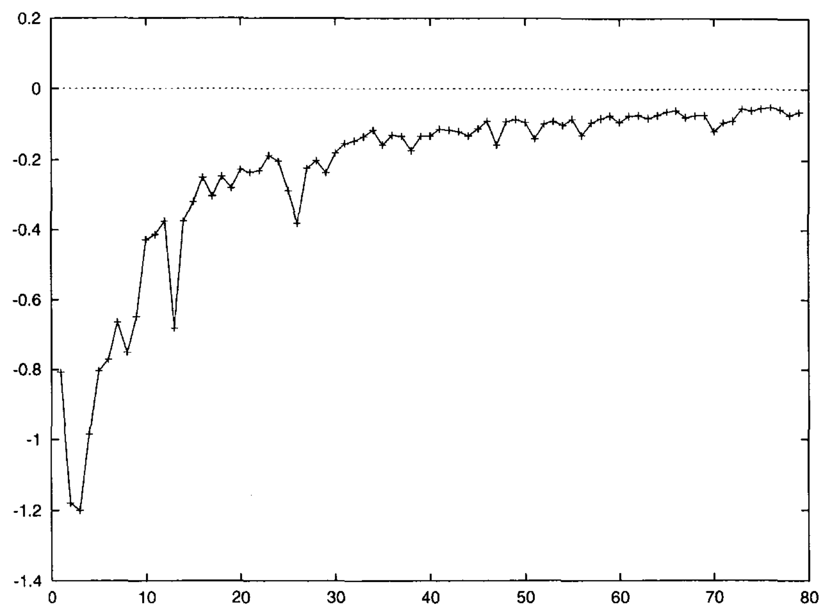


Fig. 5. Time series of λ_T for $T=1, 2, \ldots, 80$ resulting from the data of Fig. 4; $N=2^{16}$.

Figure 4 depicts the time evolution of $\Delta u(t)$ for $N = 2^{16}$ units, where the initial conditions have been sampled from a uniform distribution, corresponding to $\Delta u(0) = -\frac{1}{24}$. Computer experiments for higher values of N as well as the figure suggest that in the limit N to infinity and also for the transient domain the equilibrium point $\Delta u = 0$ can be treated as a fixed-point attractor. Note that the Lyapunov-like exponent or "convergence-rate" for a homogeneous initial distribution can be directly estimated from Fig. 4. Since in the transient regime for $t > 0$ the ratio of $|\Delta u(t+1)|$ and $|\Delta u(t)|$ turns out to be roughly $\frac{1}{4}$ (the Lyapunov-like exponent is expected to be close to $\ln \frac{1}{4} < 0$).

For large times T, where the fluctuations are dominant, we expect λ_T to converge to zero (marginal stability), since the fluctuations are found in a small band, whose width decreases according to $1/\sqrt{N}$. The convergence of λ_T is displayed in Fig. 5, where we present the experimental λ_T inserting the computer experimental $\phi(t)$ in Eq. (7.8).

8. SUMMARY

For N uncoupled logistic maps with control parameter $r = 4$, a closure of the hierarchy of our resulting equations has been achieved by an approach intimately related to the classical moment problem. For finite N we approximated the time-dependent fluctuations of the equilibrium probability density (calculated exactly within the model) in terms of the mean value and its variance.

We further showed that the fluctuations in our *uncoupled* system can be explained in terms of a quasi-fixed point attractor with size dependent Gaussian-like fluctuations (marginal stability). The formalism could be applied to other mappings and also be generalized to other values of the control parameter r, at least where a smooth asymptotic limiting distribution exists. Note however, that even in this case the asymptotic values of the moments are in general not known analytically.

Another intriguing aspect would be the derivation of a pure statistical approach to the problem. Based on the evolution equation for probability densities such as the Frobenius–Perron equation,[11] one might derive the time evolution of all the variances for ($N \to \infty$). Such a solution characterizing the convergence to the fixed-point attractor would—in the absence of Gaussian fluctuations—allow a suitable closure of the hierarchy at each truncation level.

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REFERENCES

1. K. Kaneko, *Physica D* **34**:1 (1989), and references therein.
2. K. Kaneko, *Phys. Rev. Letters* **65**(12):1391 (1990).
3. H. Chaté and P. Manneville, *Chaos* **2**(3):307 (1992).
4. L. A. Bunimovich and Ya. G. Sinai, *Nonlinearity* **1**:491 (1989).
5. J. M. Hourik and I. Webman, *Phys. Rev. A* **41**:4210 (1990).
6. C. E. Campbell, K. E. Kürten, and E. Krotscheck, *Phys. Rev. B* **25**:1633 (1982).
7. D. Welcomme, *Mémoire de Licence* (Dep. of Physics, University of Brussels, 1996).
8. N. G. van Kampen, *Stochastic processes in physics and chemistry* (North-Holland Publishing Company, Amsterdam/New York/Oxford, 1981).
9. P. Grassberger and I. Procaccia, *Physica D* **189** (1983).
10. L. Bunimovich and M. Jiang, *Chaos* **7**(1):21 (1997).
11. A. Lasota and M. Mackey, *Probabilistic Description of Deterministic Systems* (Cambridge University Press, Cambridge, 1985).